Coordinate Representations of Vectors

- Let $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be a finite, ordered basis of a vector space V. Any vector $\mathbf{v} \in V$ can be written uniquely as

$$\alpha_1 \mathbf{x}_1 + \cdots + \alpha_n \mathbf{x}_n.$$

  The vector $[\mathbf{v}]_{\mathcal{B}} = \langle \alpha_1, \ldots, \alpha_n \rangle \in \mathbb{R}^n$ is called the **coordinate representation** of $\mathbf{v}$ with respect to the ordered basis $\mathcal{B}$.
- If V is an n-dimensional vector space and $\mathcal{B}$ is any ordered basis of V, then coordinate representation gives an isomorphism from V to $\mathbb{R}^n$.

Transition Matrices

Let V be a finite dimensional vector space.
Let $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $\mathcal{C} = \{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$ be bases of V.
Let $\mathrm{id} : V \to V$ be the identity function.

- The **transition matrix** matrix $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}$ is the $n \times n$ matrix whose j^{th} column is the vector $[\mathbf{x}_j]_{\mathcal{C}}$.
- **Theorem 4.26.1:** For all $\mathbf{x} \in V$, we have $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}}$.

$$\begin{tikzcd} V \arrow[r, "\mathrm{id}"] \arrow[d, "{[\,\cdot\,]_{\mathcal{B}}}"'] & V \arrow[d, "{[\,\cdot\,]_{\mathcal{C}}}"] \\ \mathbb{R}^n \arrow[r, "{[\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}}"] & \mathbb{R}^n \end{tikzcd}$$

- **Theorem 4.26.2:** The matrix $[\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}$ is invertible, and $([\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}})^{-1} = [\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}}$.

Matrix Representations of Linear Transformations

Let V and W be finite dimensional vector spaces.
Let $\mathcal{B} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ be an ordered basis of V, and let $\mathcal{C} = \{\mathbf{y}_1, \ldots, \mathbf{y}_m\}$ be an ordered basis of W.
Let $f : V \to W$ be a linear transformation.

- We define $[f]_{\mathcal{B}}^{\mathcal{C}}$ to be the matrix whose columns are $[f(\mathbf{x}_1)]_{\mathcal{C}}$, $[f(\mathbf{x}_2)]_{\mathcal{C}}, \ldots [f(\mathbf{x}_n)]_{\mathcal{C}}$.
- **Theorem 4.33:** With the above notation, for all $\mathbf{x} \in V$, we have

$$[f]_{\mathcal{B}}^{\mathcal{C}}[\mathbf{x}]_{\mathcal{B}} = [f(\mathbf{x})]_{\mathcal{C}}.$$

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "{[\,\cdot\,]_{\mathcal{B}}}"'] & W \arrow[d, "{[\,\cdot\,]_{\mathcal{C}}}"] \\ \mathbb{R}^n \arrow[r, "{[f]_{\mathcal{B}}^{\mathcal{C}}}"] & \mathbb{R}^m \end{tikzcd}$$

Transition Matrices

Theorem 4.35: Let V be a finite dimensional vector space, with ordered bases $\mathcal{B}$ and $\mathcal{C}$. Let $f : V \to V$ be a linear transformation, and let $P = [\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}}$. Then

$$[f]_{\mathcal{C}}^{\mathcal{C}} = P^{-1}[f]_{\mathcal{B}}^{\mathcal{B}}P.$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[d, "{[f]_{\mathcal{B}}^{\mathcal{B}}}"'] & \mathbb{R}^n \arrow[l, "{P = [\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}}}"'] \arrow[d, "{[f]_{\mathcal{C}}^{\mathcal{C}}}"] \\ \mathbb{R}^n \arrow[r, "{P^{-1} = [\mathrm{id}]_{\mathcal{B}}^{\mathcal{C}}}"'] & \mathbb{R}^n \end{tikzcd}$$